Lemma 1 (a corollary of Littlewood's Principle).
Let $E \in \mathcal{M}$ with $m(E) < +\infty$, and let $\varepsilon > 0$. Then $\exists$ a step-function φ on $\mathbb{R}$ vanishing off on a bounded interval such that
$$\chi_E - \varphi = 0 \text{ on } \mathbb{R} \setminus A$$
for some A with $m(A) < \varepsilon$. $(\because m(E) < +\infty)$

Pf. By Littlewood's 1st principle, $\exists$ $U := I_1 \cup \cdots \cup I_N$ (disjoint open intervals $I_1, \cdots, I_n$) s.t. $m(E \triangle U) < \varepsilon$. Since
$$U \subseteq E \cup (U \setminus E) \subseteq E \cup (E \triangle U)$$
it follows that $m(U) = \sum_{i=1}^{N} m(I_i) < +\infty$ & so one can take a finite-length interval $(a,b) \supseteq I_i$ $\forall i$, and define
$$\psi := \chi_U, \text{ i.e. } \psi(x) = \begin{cases} 1 & \text{on } U \\ 0 & \text{outside } U \end{cases}$$
Then $\psi = \chi_E$ except on $U \triangle E$ which is of measure $< \varepsilon$. Since $U \subseteq (a,b)$, $\psi = 0$ outside the finite interval (a,b), i.e. $\psi(x) = 0 \ \forall x \in (-\infty, a] \cup [b, \infty)$.

Proposition 1. Let $E_i \in \mathfrak{M}$ with $m(E_i) < +\infty$ $\forall\, i = 1, 2, \cdots, N$ and $f := \sum_{i=1}^{N} c_i \chi_{E_i}$ with each $c_i \in \mathbb{R}$. Then, for any $\varepsilon > 0$, $\exists$ a step-function $\varphi : \mathbb{R} \to \mathbb{R}$ vanishing outside a finite interval and $A \in \mathfrak{M}$ with $m(A) < \varepsilon$ such that

$$f = \varphi \quad \text{on } \mathbb{R} \setminus A .$$

Proof. By Lemma 1, $\exists\, U_i$ (a union of finitely many disjoint open intervals contained in a finite interval (a_i, b_i)) such that $m(E_i \triangle U_i) < \frac{\varepsilon}{N}$.
Let $\varphi := \sum_{i=1}^{N} c_i \chi_{U_i}$ and $A = \bigcup_{i=1}^{N} (E_i \triangle U_i)$.

Then $m(A) < \varepsilon$ and

$$f = \varphi \quad \text{on } \mathbb{R} \setminus A .$$

Appendix 1. Let $m^*(E) < +\infty$. Then $\circlearrowleft$ =

(parts of Littlewood 1st principle)

(ii) E is outer-regular (equivalently $E \in \mathfrak{M}$)

(vi) $\forall \varepsilon > 0 \ \exists \ U := \overset{n}{\underset{i=1}{\dot\bigcup}} I_i$ with disjoint open intervals $I_1, \dots I_n$ s.t.

$$m^*(E \triangle U) < \varepsilon.$$

Pf (vi) $\Rightarrow$ (ii) (true even $m^*(E) = +\infty$). Let $\varepsilon > 0$. Take U as in (vi). Since

$$\varepsilon > m^*(E \triangle U) \geq \inf\{ m(G) : \text{open } G \supseteq E \triangle U \}$$

$\exists$ open $G_\varepsilon \supseteq E \triangle U \ (\supseteq E \setminus U)$ s.t. $m^*(G_\varepsilon) < \varepsilon$.

Then $\exists$ $G := U \cup G_\varepsilon \supseteq E$ and

$$G \setminus E \subseteq (U \setminus E) \cup G_\varepsilon \text{ of outer-mea } < \varepsilon + \varepsilon = 2\varepsilon$$

so (ii) holds.

(ii) $\Rightarrow$ (vi). Let $\varepsilon > 0$. By (ii) $\exists$ open $G \supseteq E$ s.t.

$$m^*(G \setminus E) < \frac{\varepsilon}{2} \left(\text{so } \underset{m(G)}{\overset{\|}{m^*(G)}} = m^*\big((G \setminus E) \cup E\big) < \frac{\varepsilon}{2} + m^*(E) < +\infty \right)$$

By the structure theorem for open sets, G can be expressed as a disjoint union of countably many open intervals I_n ($n \in \mathbb{N}$) so

$$\sum_{n=1}^{\infty} \ell(I_n) = \sum_{n=1}^{\infty} m(I_n) = m(G) < +\infty$$

and hence $\exists$ $N \in \mathbb{N}$ s.t.

$$\sum_{n=N+1}^{\infty} \ell(I_n) < \frac{\varepsilon}{2}.$$

Let $U := \bigcup_{n=1}^{N} {}_{\circ} I_n$. Then

$$\begin{aligned} E \triangle U &= (E \setminus U) \cup (U \setminus E) \\ &\subseteq (G \setminus U) \cup (G \setminus E) \\ &\subseteq \underbrace{\bigcup_{n=N+1}^{\infty} I_n}_{\text{of mea} < \frac{\varepsilon}{2}} \cup \underbrace{(G \setminus E)}_{\text{of outer mea} < \frac{\varepsilon}{2}} \end{aligned}$$

so $m^*(E \triangle U) < \varepsilon$, showing (vi).

Definition (inner measure). Let $A \subseteq \mathbb{R}$. Define

$$m_*(A) := \sup\{ m^*(F) : \text{closed } F \subseteq A \}$$

same as $m(F)$

Proposition. Suppose $E \in \mathcal{M}$. Then

$$m_*(E) = m(E) = m^*(E)$$

proof. Let $\varepsilon > 0$. By Littlewood's 1st principle, $\exists$ closed F_ε & open G_ε with $F_\varepsilon \subseteq E \subseteq G_\varepsilon$ s.t.

$m(E \setminus F_\varepsilon) < \varepsilon$ and $m(G_\varepsilon \setminus E) < \varepsilon$. Hence ~~$m(G_\varepsilon \setminus F_\varepsilon) < 2\varepsilon$,~~

$$m(E\setminus F_\varepsilon) = m(E) - m(F_\varepsilon) \qquad m(G_\varepsilon \setminus E) = m(G_\varepsilon) - m(E)$$

$$m^*(E) \leq m(G_\varepsilon) = m(G_\varepsilon \setminus E) + m(E) \leq \varepsilon + m(E)$$
$$= \varepsilon + \left(m(E \setminus F) + m(F)\right) \leq 2\varepsilon + m(F) \leq 2\varepsilon + m_*(E),$$

valid $\forall\, \varepsilon > 0$, so $m^*(E) \leq m_*(E)$ and hence the equality holds (why?)

Proposition 2. Let $E \subseteq \mathbb{R}$ be s.t

$$m_*(E) = m^*(E) < +\infty \qquad (*)$$

Then $E \in \mathfrak{M}$. (very important that $m^*(E) < +\infty$).

proof For each $n \in \mathbb{N}$ take closed F_n & open G_n with

$F_n \subseteq E \subseteq G_n$ s.t. $m(G_n) < m^*(E) + 1/2n$, $m(F_n) > m_*(E) - 1/2n$ (def of m_* & m^*)

and it follows from $(*)$ that $m(G_n) - m(F_n) < 1/n$ so

$m(H) - m(K) = 0$ where $H = \bigcap_{n \in \mathbb{N}} G_n$, $K = \bigcup_{n \in \mathbb{N}} F_n \in \mathcal{B} \subseteq \mathfrak{M}$

with $K \subseteq E \subseteq H$. Since $H \setminus E \subseteq H \setminus K$ of measure zero

it follows that $E \in \mathfrak{M}$.

2nd Method of Pf (in terms of limits rather than "order")

Take a seq (G_n) of open sets containing E s.t. $\lim_n m(G_n) = m^*(E)$

& ... (F_n) ... closed sets contained in E s.t. $\lim_n m(F_n) = m_*(E)$

Then $\lim_n m(G_n) = \lim_n m(F_n) < +\infty$ by the important assumption

$(*)$ so, letting $H := \bigcap_{n=1}^{\infty} G_n \in \mathfrak{M}$, $K := \bigcup_{n=1}^{\infty} F_n \in \mathfrak{M}$, it follow

that $K \subseteq E \subseteq H$ &

$$m(H\setminus K) \le \lim_K (G_n \setminus K_n) = \lim_n G_n - \lim_n K_n = 0$$

that $E \setminus K$ ($\&$ $H \setminus E$) $\in \mathfrak{M}$ (of mea. zero)

so $E \left(= K \cup (E\setminus K)\right) \in \mathfrak{M}$

Note. $\exists\, E \notin \mathfrak{M}$ with $m_*(E) = m^*(E) = +\infty$.

Take a non-measurable subset $D \subseteq (0,1)$

and let $E := D \cup [2, \infty)$ (so E not measurable)

Then $m_*(E) = +\infty = m^*(E)$.